# On Riemannian nonsymmetric spaces and flag manifolds

Abdelkader Bouyakoub *
Michel Goze †
Elisabeth Remm ‡

February 2, 2008

**2000 Mathematics Subject Classification.** 53C20. 53C35

**Keywords.** $\Gamma$-symmetric spaces. Adapted riemannian metrics. Graded Lie algebras.

**Abstract**

In this work we study riemannian metrics on flag manifolds adapted to the symmetries of these homogeneous nonsymmetric spaces(. We first introduce the notion of riemannian $\Gamma$-symmetric space when $\Gamma$ is a general abelian finite group, the symmetric case corresponding to $\Gamma = \mathbb{Z}_2$. We describe and study all the riemannian metrics on $SO(2n+1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(2n+1-r_1-r_2-r_3)$ for which the symmetries are isometries. We consider also the lorentzian case and give an example of a lorentzian homogeneous space which is not a symmetric space.

## 1 Introduction

If $M$ is a homogeneous symmetric space, then at each point $x \in M$ we have a symmetry $s_x$ that is a diffeomorphim of $M$ satisfying $s_x^2 = Id$. It is equivalent to say that at every point $x \in M$ we have a subgroup $\Gamma_x$ of *Diff(M)* isomorphic to $\mathbb{Z}_2$. The notion of $\Gamma$-symmetric space is a generalization of the classical notion of symmetric space by considering a general finite abelian group of symmetries $\Gamma$ instead of $\mathbb{Z}_2$. The case $\Gamma = \mathbb{Z}_k$ was considered from the algebraic point of view by V. Kac and the differential geometric approach was carried

*Bouyakoub Université d'Oran Es-Sénia, Institut de Mathématiques, BP 1524, El Menawer 3100, ORAN, ALGERIE

†M.Goze@uha.fr, Université de Haute Alsace, F.S.T., 4, rue des Frères Lumière - 68093 MULHOUSE - FRANCE

‡corresponding author: E.Remm@uha.fr, Université de Haute Alsace, F.S.T., 4, rue des Frères Lumière - 68093 MULHOUSE - FRANCE. This work has been supported for the three authors by AUF project MASI 2005-2006

out by A.J. Ledger, M. Obata [6] and O. Kowalski [3] in terms of $k$-symmetric spaces. A $k$-manifold is a homogeneous reductif space and the classification of these varieties is given by the corresponding classification of Lie algebras. The general notion of $\Gamma$-symmetric spaces was introduced by R. Lutz [5] and was algebraically reconsidered by Y. Bahturin and M. Goze [1]. In this last work the authors proved, in particular, that a $\Gamma$-symmetric space is a homogeneous space $G/H$ and the Lie algebra $\mathfrak{g}$ of $G$ is $\Gamma$-graded. They give also a classification of $\Gamma$-symmetric spaces when $G$ is a classical simple complex Lie algebra and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. We can see in particular that the flag manifold admits such a structure. The particular case of Grassmannian manifolds comes into the framework of symmetric manifolds. But for a general flag manifold, it is not the case. There is a great interest to study these manifolds, in an affine or riemannian point of view. For example, in loops groups theory we have to look complex algebraic homogeneous spaces $U_n$ and these spaces are Grassmannians or flag manifolds. We will describe symmetries which provide a flag manifold with a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure. We then study riemannian metrics adapted to this structure, that is riemannian metrics for which the riemannian connection is the canonical torsion free connection of a homogeneous space. We have to impose in addition that the symmetries are isometries (in the case of riemannian symmetric spaces this is a natural consequence of the very definition) We compute these metrics for flag manifolds and describe the associated riemannian invariants in some peculiar cases.

## 2 $\Gamma$-symmetric spaces

In this section we recall some basical notions (see [1] for more details).

### 2.1 Definition

Let $\Gamma$ be a finite abelian group. A $\Gamma$-symmetric space is a triple $(G, H, \Gamma_G)$ where $G$ is a connected Lie group, $H$ a closed subgroup of $G$ and $\Gamma_G$ an abelian finite subgroup of the group of automorphisms of $G$ isomorphic to $\Gamma$:

$$\Gamma_G = \{\rho_\gamma \in Aut(G),\ \gamma \in \Gamma\}$$

such that $H$ lies between $G_\Gamma$ the closed subgroup of $G$ consisting of all elements left fixed by the automorphisms of $\Gamma_G$ and the identity component of $G_\Gamma$. The elements of $\Gamma_G$ satisfy :

$$\begin{cases} \rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1 \gamma_2}, \\ \\ \rho_e = Id \ \text{ where e is the unit element of } \ G, \\ \\ \rho_\gamma(g) = g \ , \forall \gamma \in \Gamma \Longleftrightarrow g \in H. \end{cases}$$

We also suppose that $H$ does not contain any proper normal subgroup of $G$.

## 2.2 $\Gamma$-symmetries on the homogeneous space $M = G/H$

Given a $\Gamma$-symmetric space $(G, H, \Gamma_G)$ we construct for each point $x$ of $M = G/H$ a subgroup $\Gamma_x$ of *Diff*$(M)$, the group of diffeomorphisms of $M$, isomorphic to $\Gamma$ which has $x$ as an isoled fixed point. We denote by $\bar{g}$ the class of $g \in G$ in $M$ and $e$ the identity of $G$. We consider

$$\Gamma_{\bar{e}} = \{s_{(\gamma,\bar{e})} \in \mathit{Diff}(M),\ \gamma \in \Gamma\}$$

with $s_{(\gamma,\bar{e})}(\bar{g}) = \overline{\rho_\gamma(g)}$.

In another point $x = \overline{g_0}$ of $M$ we have

$$\Gamma_x = \{s_{(\gamma,x)} \in \mathit{Diff}(M),\ \gamma \in \Gamma\}$$

with $s_{(\gamma,\bar{g_0})}(y) = g_0(s_{(\gamma,\bar{e})})(g_0^{-1}y)$. All these subgroups $\Gamma_x$ of *Diff*$(M)$ are isomorphic to $\Gamma$.

Since for every $x \in M$ and $\gamma \in \Gamma$, the map $s_{(\gamma,x)}$ is a diffeomorphism of $M$, such that $s_{(\gamma,x)}(x) = x$ the tangent linear map $(Ts_{(\gamma,x)})_x$ is in $GL(T_xM)$. Thus, for every $x \in M$, we obtain a linear representation

$$S_x : \Gamma \longrightarrow GL(T_xM)$$

defined by

$$S_x(\gamma) = (Ts_{(\gamma,x)})_x$$

and $S(\gamma)$ can be considered as a $(1,1)$-type tensor on $M$ satisfying

1. For every $\gamma \in \Gamma$, the map $x \in M \longrightarrow S_x(\gamma)$ is of class $\mathcal{C}^\infty$,
2. For every $x \in M$, $\{X_x \in T_x(M)$ such that $S_x(\gamma)(X_x) = X_x,\ \forall \gamma\} = \{0\}$.

If we denote by

$$\check{\Gamma}_x = \{S_x(\gamma), \gamma \in \Gamma\}$$

then $\check{\Gamma}_x$ is a subgroup of $GL(n, T_x(M))$ isomorphic to $\Gamma$.

## 2.3 $\Gamma$-grading of the Lie algebra of $G$

Let $\mathfrak{g}$ be the Lie algebra of $G$. Each automorphism $\rho_\gamma$ of $G$ induces an automorphism $\tau_\gamma$ of $\mathfrak{g}$. Let $\check{\Gamma}$ be the set of all these automorphisms $\tau_\gamma$. Then $\check{\Gamma}$ is a finite abelian subgroup of $Aut(\mathfrak{g})$ isomorphic to $\Gamma$ and $\mathfrak{g}$ is graded by $\Gamma$ that is

$$\mathfrak{g} = \oplus_{\gamma \in \Gamma}\ \mathfrak{g}_\gamma$$

with $[\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\gamma_2}] \subset \mathfrak{g}_{\gamma_1\gamma_2}$. The group $\check{\Gamma}$ is canonically isomorphic to the dual group of $\Gamma$. Conversely every $\Gamma$-grading of $\mathfrak{g}$ defines a $\Gamma$-symmetric space $(G, H, \Gamma_G)$ where $G$ is a Lie group corresponding to $\mathfrak{g}$ and the Lie algebra of $H$ is the component $\mathfrak{g}_e$ corresponding to the identity of $\Gamma$.

## 2.4 Canonical connections of a $\Gamma$-symmetric space

If $(G, H, \Gamma_G)$ is a $\Gamma$-symmetric space, the homogeneous space $M = G/H$ is reductive. In fact the Lie algebra $\mathfrak{g}$ being $\Gamma$-graded we have $\mathfrak{g} = \oplus\ \mathfrak{g}_\gamma = \mathfrak{g}_e \oplus \mathfrak{m}$ with $\mathfrak{m} = \oplus_{\gamma\in\Gamma,\gamma\neq e}\ \mathfrak{g}_\gamma$ and $[\mathfrak{g}_e, \mathfrak{m}] \subset \mathfrak{m}$. If we suppose $H$ connected, this last relation means that $ad(H)(\mathfrak{m}) \subset \mathfrak{m}$. If $ad(H)(\mathfrak{m}) = \mathfrak{m}$, then any connection on $G/H$ invariant by left translations of $G$ is defined by the $\mathfrak{g}_e$-component $\omega$ of the canonical 1-form $\theta$ of $G$. In this case the curvature $\Omega$ is given by

$$\Omega(X, Y) = -\frac{1}{2}[X, Y]_{\mathfrak{g}_e}$$

for every $X, Y \in \mathfrak{m}$. Moreover the Lie algebra of the holonomy group in $\bar{e}$ is generated by all elements of the form $[X, Y]_{\mathfrak{g}_e}$, $X, Y \in \mathfrak{m}$. This connection is called [4] the canonical connection of the principal fibered bundle $G(G/H, H)$. Its torsion and curvature are given at the origin $\bar{e}$ of $G/H$ by

$$T(X, Y)_{\bar{e}} = -[X, Y]_{\mathfrak{m}}$$

$$R(X, Y)_{\bar{e}} = -[X, Y]_{g_e}$$

for all $X, Y \in \mathfrak{m}$.
If $\Gamma = \mathbb{Z}/2\mathbb{Z}$, that is if $(G, H, \Gamma_G)$ is a symmetric space, then the canonical connection $\nabla$ on $M = G/H$ is torsion free. In all the other cases, for example when $\Gamma$ is the Klein group, the torsion $T$ of $\nabla$ does not vanish. We consider then the connection $\overline{\nabla}$ given by

$$\overline{\nabla} = \nabla - T.$$

This connection is torsion free. Its curvature tensor writes

$$\begin{aligned}(R_{\overline{\nabla}}(X, Y)(Z))_{\bar{e}} = \ & \tfrac{1}{4}[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{4}[Y, [X, Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \tfrac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} \\ & -[[X, Y]_{\mathfrak{g}_e}, Z]_{\mathfrak{m}}\end{aligned}$$

for all $X, Y, Z \in \mathfrak{m}$ while the curvature of $\nabla$ is given by

$$(R_\nabla(X, Y)(Z))_{\bar{e}} = -[[X, Y]_{\mathfrak{g}_e}, Z]_{\mathfrak{m}}.$$

The geodesics of $\nabla$ and $\overline{\nabla}$ are the same. The connection $\overline{\nabla}$ is called the torsion-free canonical connection. We can note that the canonical connection satisfies also

$$\begin{aligned}&\nabla T = 0\\ &\nabla R_\nabla = 0.\end{aligned}$$

Moreover the symmetries $S_x(\gamma)$ are affine transformations with respect to $\nabla$.

## 3 Riemannian Γ-symmetric spaces

### 3.1 Riemannian symmetric space

Let $M = G/H$ a homogeneous symmetric space, where $G$ is a connected Lie group. We denote by 0 the coset $H$ of $M$, that is the class on $G/H$ of the identity 1 of $G$. The Lie algebra $\mathfrak{g}$ of $G$ is $\mathbb{Z}_2$-graded

$$\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_a$$

where $\mathbb{Z}_2 = \{e, a\}$ and this decomposition is $ad(H)$-invariant. The Lie algebra of $H$ is $\mathfrak{g}_e$ and the tangent space at 0 $T_0M$ is identified to $\mathfrak{g}_a$.

Every $G$-invariant metric $g$ on $G/H$ is given by an $ad(H)$-invariant non degenerate symmetric bilinear form $B$ on $\mathfrak{g}/\mathfrak{g}_e$ by

$$B_{\mathfrak{g}}(\bar{X}, \bar{Y}) = g(X, Y)$$

for $X, Y \in \mathfrak{g}$ and $\bar{X}$ the class of $X$ in $\mathfrak{g}/\mathfrak{g}_e$. We identify $X \in \mathfrak{g}$ with the projection on $M$ of the associated left invariant vector field on $G$. Moreover $g$ is a riemannian metric if and only if $B$ is positive definite. The identification of $\mathfrak{g}/\mathfrak{g}_e$ with $\mathfrak{g}_a$ permits to consider $B$ as a non degenerate bilinear form on $\mathfrak{g}_a$. This form satisfies

$$B(X, [Y, Z]_{\mathfrak{g}_a}) = B(X, 0)$$

for all $Y, Z \in \mathfrak{g}_a$ because $[\mathfrak{g}_a, \mathfrak{g}_a] \subset \mathfrak{g}_e$. Then $B(X, [Y, Z]_{\mathfrak{g}_a}) + B([Y, X], Z) = 0$ for all $X, Y, Z \in \mathfrak{m} = \mathfrak{g}_a$ and $M = (G/H, g)$ is naturally reductive. This means that the riemannnian connection of $G$ coincides with the canonical torsion free connection $\overline{\nabla}$ of $M$ and the symmetries $S_x \in \Gamma_x$ for all $x \in M$ are isometric. Conversely let $g$ be a metric on $G/H$ such that for each $x \in M$ $S_x$ is an isometry. If $ad(H)$ is a compact subgroup of $GL(\mathfrak{g})$, then there exists an $ad(H)$-invariant inner product $\tilde{B}$ on $\mathfrak{g}$ such that

1)$\tilde{B}(\mathfrak{g}_e, \mathfrak{g}_a) = 0$
2)$\tilde{B} \mid_{\mathfrak{g}_a} = B$ induces the riemannian metric $g$ on $G/H$

Since $[\mathfrak{g}_a, \mathfrak{g}_a] \subset \mathfrak{g}_e$ the naturally reductivity is obvious and the riemannian connection coincides with $\overline{\nabla}$.

Recall that if $G$ is a semi-simple Lie group then $B$ is neither but the restriction to $\mathfrak{g}_a$ of the Killing-Cartan form $\tilde{B}$ on $G$ that is

$$B(X, Y) = tr(adX \circ adY)$$

for all $X, Y \in \mathfrak{m}$.

### 3.2 Riemannian Γ-symmetric spaces

Let $\Gamma$ be a finite abelian group not isomorphic to $\mathbb{Z}_2$ and $g$ any $G$-invariant metric on a $\Gamma$-symmetric space $M = G/H$. Let us quppose that the symmetries $S_x$ are isometries for $g$. As $\Gamma$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, this property doesn't imply in general the coincidence of the associated Levi-Civita connection and $\overline{\nabla}$.

**Definition 1** *Let $(G, H, \Gamma_G)$ be a $\Gamma$-symmetric space and $g$ a $G$-invariant metric on $M$. We say that $(M, g)$ is a riemannian $\Gamma$-symmetric space if the symmetries $S_x$ are isometries for all $x \in M$.*

**Lemma 2** *Let $(G, H, \Gamma_G)$ a $\Gamma$-symmetric space and $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ the associated $\Gamma$-grading of the Lie algebra $\mathfrak{g}$ of $G$. Then for every $\gamma \in \Gamma$*

$$ad(H)\mathfrak{g}_\gamma \subset \mathfrak{g}_\gamma$$

*Proof.* Let $X$ be in $\mathfrak{g}_\gamma$. For every $\tau_\alpha \in \check{\Gamma}$, we have $\tau_\alpha(X) = \lambda(\gamma, \alpha)X$ with $\lambda(\gamma, \alpha) = \pm 1$. Then

$$\tau_\alpha(ad(h)(X)) = ad(\rho_\alpha(h))(\tau_\alpha(X)) = \lambda(\gamma, \alpha) ad(h)(X)$$

because all the elements of $H$ are invariant by the automorphisms $\rho_\alpha$. This proves that $ad(h)X \in \mathfrak{g}_\gamma$.

**Proposition 3** *If $ad(H)$ is a compact subgroup of $GL(\mathfrak{g})$ and $g$ a $G$-invariant metric on the $\Gamma$-symmetric space $M = G/H$ then there exits an $ad(H)$-inner product $\tilde{B}$ on $\mathfrak{g}$ such that*

1) $\tilde{B}(\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}) = 0$ for $\gamma \neq \gamma'$ in $\Gamma$
2) $\tilde{B} \mid_{\mathfrak{g}_a} = B$ induces the riemannian metric $g$ on $G/H$

*Proof.* Since each homogeneous component $\mathfrak{g}_\gamma$ is invariant by $ad(H)$, there exists an inner product $B$ on $\mathfrak{g}$ which is $ad(\mathfrak{g}_e)$-invariant and which defines $g$. As the symmetries $s_(\gamma, x)$ are isometries, we deduce that the automorphisms $\tau_\gamma$ are isometries for $\tilde{B}$. If $X \in \mathfrak{g}_\gamma, Y \in \mathfrak{g}_{\gamma'}$, there exits $\alpha \in \Gamma$ such that

$$\tau_\alpha(X) = \lambda(\alpha, \gamma)X, \ \ \tau_\alpha(Y) = \lambda(\alpha, \gamma')Y$$

with $\lambda(\alpha, \gamma)\lambda(\alpha, \gamma') = -1$. Thus

$$\tilde{B}(X, Y) = \tilde{B}(\tau_\alpha(X), \tau_\alpha(Y)) = -\tilde{B}(X, Y) \text{and } \tilde{B}(X, Y) = 0.$$

**Example.** Let us consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space

$$(S0(5); SO(2) \times SO(2) \times SO(1), \Gamma_G)$$

where $\Gamma_G$ is defined as follows. One writes a general element of $so(5)$ by

$$so(5) = \left\{ \begin{pmatrix} 0 & x_1 & a_1 & a_2 & b_1 \\ -x_1 & 0 & a_3 & a_4 & b_2 \\ -a_1 & -a_3 & 0 & x_2 & c_1 \\ -a_2 & -a_4 & -x_2 & 0 & c_2 \\ -b_1 & -b_2 & -c_1 & -c_2 & 0 \end{pmatrix}, x_i, a_i, b_i, c_i \in \mathbb{R} \right\}.$$

We put

$$\begin{aligned}\mathfrak{g}_e &= \{X \in so(5) \,/\, a_i = b_i = c_i = 0\}\,,\\ \mathfrak{g}_a &= \{X \in so(5) \,/\, x_i = b_i = c_i = 0\}\,,\\ \mathfrak{g}_b &= \{X \in so(5) \,/\, x_i = b_i = c_i = 0\}\,,\\ \mathfrak{g}_c &= \{X \in so(5) \,/\, x_i = a_i = b_i = 0\}\,.\end{aligned}$$

If $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, a, b, c\}$, then $so(5) = \mathfrak{g}_e \oplus \mathfrak{g}_a \oplus g_b \oplus g_c$ is a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading. In this case

$$\check{\Gamma} = \{\tau_e, \tau_a, \tau_b, \tau_c\}$$

with $\tau_e = id, \tau_a(X) = X$ for $X \in \mathfrak{g}_e \oplus \mathfrak{g}_a$, $\tau_a(X) = -X$ for $X \in \mathfrak{g}_b \oplus \mathfrak{g}_c$, $\tau_b(X) = X$ for $X \in \mathfrak{g}_e \oplus \mathfrak{g}_b$, $\tau_b(X) = -X$ for $X \in \mathfrak{g}_a \oplus \mathfrak{g}_c$ and $\tau_c(X) = X$ for $X \in \mathfrak{g}_e \oplus \mathfrak{g}_c$, $\tau_c(X) = -X$ for $X \in \mathfrak{g}_a \oplus \mathfrak{g}_b$. Since $G = SO(5)$ is connected, this grading gives a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure on $M = S0(5)/SO(2) \times SO(2) \times SO(1)$ and $\mathfrak{g}_e$ is the Lie algebra of $H = SO(2) \times SO(2) \times SO(1)$. We denote by $\{\{X_1, X_2\}, \{A_1, A_2, A_3, A_4\}, \{B_1, B_2\}, \{C_1, C_2\}\}$ the basis of $so(5)$ where each big letter corresponds to the matrix of $so(5)$ with the small letter equal to 1 and other coefficients are zero. This basis is adapted to the grading. Let us denote by $\{\omega_1, \omega_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \gamma_1, \gamma_2\}$ the dual basis.

*Every $ad(H)$-invariant symmetric bilinear form $B$ on $\mathfrak{m} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ such that $B(\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}) = 0$ for $\gamma \neq \gamma'$ in $\Gamma$ is written*

$$B = t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1\alpha_4 - \alpha_2\alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2).$$

In fact, since $H$ is connected the bilinear product $B$ is $ad(H)$-invariant if and only if

$$B([X,Y], Z) + B(Y, [X,Z]) = 0$$

for $Y, Z \in \mathfrak{m}$ and $X \in \mathfrak{h} = \mathfrak{g}_e$.

The brackets of $so(5)$ with respect to the basis $\{X_i, A_i, B_i, C_i\}$ are summarized in the follwing table

| | $X_1$ | $X_2$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $B_1$ | $B_2$ | $C_1$ | $C_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | $0$ | $0$ | $-A_3$ | $-A_4$ | $A_1$ | $A_2$ | $-B_2$ | $B_1$ | $0$ | $0$ |
| $X_2$ | | $0$ | $-A_2$ | $A_1$ | $-A_4$ | $A_3$ | $0$ | $0$ | $-C_2$ | $C_1$ |
| $A_1$ | | | $0$ | $-X_2$ | $-X_1$ | $0$ | $-C_1$ | $0$ | $B_1$ | $0$ |
| $A_2$ | | | | $0$ | $0$ | $-X_1$ | $-C_2$ | $0$ | $0$ | $B_1$ |
| $A_3$ | | | | | $0$ | $-X_2$ | $0$ | $-C_1$ | $B_2$ | $0$ |
| $A_3$ | | | | | | $0$ | $0$ | $-C_2$ | $0$ | $B_2$ |
| $B_1$ | | | | | | | $0$ | $-X_1$ | $-A_1$ | $-A_2$ |
| $B_2$ | | | | | | | | $0$ | $-A_3$ | $-A_4$ |
| $C_1$ | | | | | | | | | $0$ | $-X_2$ |
| $C_2$ | | | | | | | | | | $0$ |

The identity $B([X_i, A_j], A_j) = 0$ implies

$$B(A_1, A_3) = B(A_1, A_2) = B(A_2, A_4) = B(A_3, A_4) = 0,$$

$$B(B_1, B_2) = B(C_1, C_2) = 0.$$

The identity $B([X_2, A_i], A_j) + B(A_i, [X_2, A_j]) = 0$ gives for $i \neq j$

$$\begin{array}{l} B(A_2, A_3) + B(A_1, A_4) = 0, \\ -B(A_3, A_3) + B(A_1, A_1) = 0 \\ -B(A_4, A_4) + B(A_2, A_2) = 0 \\ -B(A_2, A_2) + B(A_1, A_1) = 0 \end{array}$$

In the same way we find

$$\begin{array}{l} B(B_1, B_1) = B(B_2, B_2), \\ B(C_1, C_1) = B(C_2, C_2) \end{array}$$

this gives

$$B = t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1\alpha_4 - \alpha_2\alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2).$$

*The metric $g$ on*

$$S0(5)/SO(2) \times SO(2) \times SO(1)$$

*associated to $B$ is naturally reductive if and only if $t = v = w$ and $u = 0$.*
In fact, if $\mathfrak{g}$ is naturally reductive then $B$ satisfies

$$B(X, [Z, Y]_m) + B([Z, X]_{\mathfrak{m}}, Y) = 0$$

for every $X, Y, Z \in m$. In particular $B(A_1, [B_2, C_2]_{\mathfrak{m}}) + B([C_2, A_1]_{\mathfrak{m}}, B_2) = 0$ gives $-B(A_1, A_4) + B(0, B_2) = 0$ and $u = 0$. Similarly $B(A_1, [B_1, C_1]) + B([B_1, A_1], C_1) = 0$ gives $-B(A_1, A_1) + B(C_1, C_1) = 0$ that is $t = w$, and $B(B_1, [A_1, C_1]) + B([A_1, B_1], C_1) = 0$ gives $B(B_1, B_1) - B(C_1, C_1) = 0$ that is $v = w$.

**Proposition 4** *The riemannian connection $\nabla_g$ of the metric $g$ on $S0(5)/SO(2) \times SO(2) \times SO(1)$ coincides with the canonical torsion free connection $\overline{\nabla}$ if and only if $B = \sum_{i=1}^4 \alpha_i^2 + \sum_{i=1}^2 \beta_i^2 + \sum_{i=1}^2 \gamma_i^2$.*

*Remark* If $g$ is a $G$-invariant metric on $G/H$ such that its connection $\nabla_g$ is equal to $\overline{\nabla}$ the bilinear form $B$ is naturally reductive. In the previous example, since $G$ is a simple Lie group, this inner product $B$ is the restriction to $\mathfrak{m}$ of the Kiling-Cartan form $K$ of $G$.

$$B(X, Y) = K(X, Y) = tr(adX \circ adY).$$

Then the homogeneous component $\mathfrak{g}_\gamma$ are pairwise orthogonal and the $\tau_\gamma$ are isometries. But it is not the case in general.

Let us return to the general case.

**Definition 5** *Let $(G, H, \Gamma_G, g)$ a riemannian $\Gamma$-symmetric space. We say that $g$ is adapted to the $\Gamma$-structure if the Levi-Civita connection coincides with the canonical one.*

**Proposition 6** *Every riemannian $\Gamma$-symmetric space with adapted riemannian connection is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{m}$ with $\mathfrak{m} = \oplus_{\Gamma \neq e} \mathfrak{g}_\gamma$.*

*Proof.* Any $G$-invariant riemannian metric $g$ on a reductive homogeneous space $G/H$ with an $ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{m}$ corresponds to an $ad(H)$-invariant non degenerate symmetric bilinear form $B_\mathfrak{m}$ on $\mathfrak{m}$. Since $M = G/H$ is a riemannian $\Gamma$-symmetric space, its $G$-invariant riemannian metric $g$ is parallel with respect to the canonical torsionless connection $\overline{\nabla}$. Then from [4] Theorem 3.3 the riemannian connnection of $g$ and $\overline{\nabla}$ coincides on $G/H$ if and only if $B_\mathfrak{m}$ satisfies

$$B_\mathfrak{m}(X, [Y, Z]_m) + B_\mathfrak{m}([Y, Z]_\mathfrak{m}, X) = 0$$

for all $X, Y, Z \in \mathfrak{m}$. This means that $(G/H, g)$ is naturally reductive.

## 3.3 Irreducible riemannian $\Gamma$-symmetric spaces

Let $(G, H, \Gamma_G)$ a $\Gamma$-symmetric space. Since $G/H$ is a reductible homogeneous space with an $ad\,H$ invariant decomposition $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{m}$ then the Lie algebra of the holonomy group of $\nabla$ is spanned by the endomorphisms of $\mathfrak{m}$ given by $R(X, Y)_0$ for all $X, Y \in \mathfrak{m}$. Recall that $(R(X, Y)Z)_0 = -[[X, Y]_\mathfrak{h}, Z]$ for all $X, Y, Z \in \mathfrak{m}$. In particular we have $R(X, Y)_0 = 0$ as soon as $X \in \mathfrak{g}_\gamma, Y \in {}_-\gamma'$ with $\gamma, \gamma' \neq e$. For example if $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ and $R(\mathfrak{g}_a, \mathfrak{g}_b)_0 = R(\mathfrak{g}_a, \mathfrak{g}_c)_0 = R(\mathfrak{g}_b, \mathfrak{g}_c)_0 = 0$.

**Lemma 7** *Let $\mathfrak{g}$ is a simple Lie algebra $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded. Then*

$$[\mathfrak{g}_a, \mathfrak{g}_a] \oplus [\mathfrak{g}_b, \mathfrak{g}_b] \oplus [\mathfrak{g}_c, \mathfrak{g}_c] = \mathfrak{g}_e.$$

*Proof.* Let $U$ denote $[\mathfrak{g}_a, \mathfrak{g}_a] \oplus [\mathfrak{g}_b, \mathfrak{g}_b] \oplus [\mathfrak{g}_c, \mathfrak{g}_c]$. Then $I = U \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is an ideal of $\mathfrak{g}$. In fact $X \in I$ is decomposed as $X_U + X_a + X_b + X_c$. The main point is to prove that $[X_U, Y]$ is in $I$ for any $Y \in \mathfrak{g}_e$. But $X_U$ is decomposed as $[X_a, Y_a] + [X_b, Y_b] + [X_c, Y_c]$. The Jacobi identity shows that $[[X_a, Y_a], Y] \in [\mathfrak{g}_a, \mathfrak{g}_a]$. It is similary for the other components. Then $I$ is an ideal of $\mathfrak{g}$ which is simple so $U = \mathfrak{g}_e$.

Remark that in any case , as soon as $\Gamma$ is not $\mathbb{Z}_2$ the representation $ad\,\mathfrak{g}_e$ is not ireeducible on $\mathfrak{m}$. In fact each component $\mathfrak{g}_\gamma$ is an invariant subspace of $\mathfrak{m}$.

**Definition 8** *The representation $ad\,\mathfrak{g}_e$ on $\mathfrak{m}$ is called $\Gamma$-irreducible if $\mathfrak{m}$ can not be written $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with $\mathfrak{g}_e \oplus m_1$ and $\mathfrak{g}_e \oplus m_2$ are $\Gamma$-graded Lie algebras.*

**Example.** Let $\mathfrak{g}_1$ be a simple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1$. Let $\sigma_1, \sigma_2, \sigma_3$ the automorphisms of $\mathfrak{g}$ given by

$$\begin{cases} \sigma_1(X_1, X_2, X_3, X_4) = (X_2, X_1, X_3, X_4), \\ \sigma_2(X_1, X_2, X_3, X_4) = (X_1, X_2, X_4, X_3), \\ \sigma_3 = \sigma_1 \circ \sigma_2. \end{cases}$$

They define a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-graduation on $\mathfrak{g}$ and we have $\mathfrak{g}_e = \{(X,X,Y,Y)\}$, $\mathfrak{g}_a = \{(0,0,Y,-Y)\}$, $\mathfrak{g}_b = \{(X,-X,0,0)\}$ and $\mathfrak{g}_c = \{(0,0,0,0)\}$ with $X, Y \in \mathfrak{g}_1$. In particular $\mathfrak{g}_a$ is isomorphic to $\mathfrak{g}_1$ so we have $[\mathfrak{g}_e, \mathfrak{g}_a] = \mathfrak{g}_a$ and since $\mathfrak{g}_1$ is simple we can not have $\mathfrak{g}_a = \mathfrak{g}_a^1 + \mathfrak{g}_a^2$ with $[\mathfrak{g}_e, \mathfrak{g}_a^i] = \mathfrak{g}_a^i$ for $i = 1, 2$. Then $\mathfrak{g}$ is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-graded and this decomposition is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-irreducible.

Suppose now that $\mathfrak{g}$ is a simple Lie algebra. Let $K$ be the Killing-Cartan form of $\mathfrak{g}$. It is invariant by all automorphisms of $\mathfrak{g}$. In particular

$$K(\tau_\gamma X, \tau_\gamma Y) = K(X, Y)$$

for any $\tau_\gamma \in \check{\Gamma}$. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, $\alpha \neq \beta$ there exists $\gamma \in \Gamma$ such that $\tau_\gamma X = \lambda(\alpha, \gamma) X$ and $\tau_\gamma Y = \lambda(\beta, \gamma) Y$ with $\lambda(\alpha, \gamma)\lambda(\beta, \gamma) \neq 1$. Thus $K(X, Y) = 0$ and the homogeneous components $\mathfrak{g}_\gamma$ are pairewise orthogonal with respect to $K$. Moreover $K_\gamma = K|_{\mathfrak{g}_\gamma}$ is a nondegenerate bilinear form. Since $\mathfrak{g}$ is a simple Lie algebra, there exists an $ad\,\mathfrak{g}_e$-invariant inner product $\tilde{B}$ on $\mathfrak{g}$ such that the restriction $B = \tilde{B}|_\mathfrak{m}$ to $\mathfrak{m}$ defines a riemannian $\Gamma$-symmetric structure on $G/H$. This means that $\tilde{B}(\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}) = 0$ for $\gamma \neq \gamma' \in \Gamma$. We consider an orthogonal basis of $\tilde{B}$. For each $X \in \mathfrak{g}_e$, $ad\,X$ is expressed by a skew-symmetric matrix $(a_{ij}(X))$ and $K(X, X) = \sum_{i,j} a_{ij}(X) a_{ji}(X) < 0$. So $K$ is negative-definite on $\mathfrak{g}_e$.

Let $K_\gamma$ and $B_\gamma$ be the restrictions of $K$ and $B$ at the homogeneous component $\mathfrak{g}_\gamma$. Let $\beta \in \mathfrak{m}^*$ be such that

$$K_\gamma(X, Y) = B_\gamma(\beta_\gamma(X), Y)$$

for all $X, Y \in \mathfrak{g}_\gamma$ and $\beta_\gamma = \beta|_{\mathfrak{g}_\gamma}$. Since $B_\gamma$ is nondegenerate on $\mathfrak{g}_\gamma$, the eigenvalues of $\beta_\gamma$ are real and non zero. The eigenspaces $\mathfrak{g}_\gamma^1, ..., \mathfrak{g}_\gamma^p$ of $\beta_\gamma$ are pairwise orthogonal with respect to $B_\gamma$ and $K_\gamma$. But for every $Z \in \mathfrak{g}_e$ we have

$$K_\gamma([Z, X], Y) = K_\gamma(X, [Z, Y]) = B_\gamma(\beta_\gamma(X), [Z, Y])$$

so $B_\gamma(\beta_\gamma[Z, X], Y) = B_\gamma([Z, \beta_\gamma(X)], Y)$ for every $Y \in \mathfrak{g}_\gamma$ and $\beta_\gamma[Z, X] = [Z, \beta_\gamma(X)]$ that is $\beta_\gamma \circ ad\,Z = ad\,Z \circ \beta_\gamma$ for any $Z \in \mathfrak{g}_e$. This implies that $[\mathfrak{g}_e, \mathfrak{g}_\gamma^i] \subset \mathfrak{g}_\gamma^i$.

Now we shall examinate the particular case corresponding to $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. The eigenvalues of the involutive automorphisms $\tau_\gamma$ being real, the Lie algebra $\mathfrak{g}$ admits a real $\Gamma$-decomposition $\mathfrak{g} = \sum_{\gamma \in \mathbb{Z}_2 \times \mathbb{Z}_2} \mathfrak{g}_\gamma$. Then we can consider that $\mathfrak{g}$ is a real Lie algebra.

Now if $i \neq j$ then

$$K_\gamma([\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j], [\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j]) \subset K([\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j], \mathfrak{g}_e) \subset (\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j) = 0$$

and we have

$$[\mathfrak{g}_\gamma^i, \mathfrak{g}_\gamma^j] = \{0\}$$

for $i \neq j$.

**Example.** In the section 4, we study the riemannian homogeneous manifold $SO(2l+1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)$. This manifold is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric and the Lie algebra $so\,(2l+1)$ admits a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading. By referring to the study which follows we see that

$$\mathfrak{g}_a = A_1 \oplus A_2, \ \mathfrak{g}_b = B_1 \oplus B_2, \ \mathfrak{g}_c = C_1 \oplus C_2$$

with $[A_1, A_2] = [B_1, B_2] = [C_1, C_2] = 0$ and we have

$$K(A_1, A_2) = K(B_1, B_2) = K(C_1, C_2) = 0.$$

So we have an orthogonal decomposition of each invariant space $\mathfrak{g}_a, \mathfrak{g}_b, \mathfrak{g}_c$ but the graduation is $\Gamma$-irreductible. In fact we have $[A_1, B_1] = [A_2, B_2] = C_1, \ [A_1, B_2] = [A_2, B_1] = C_2$. ♣

Let $\{e, a, b, c\}$ be the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $a^2 = b^2 = c^2 = e$ and $ab = c$. Each component $\mathfrak{g}_\gamma$, $\gamma \neq e$, satisfies $[\mathfrak{g}_\gamma, \mathfrak{g}_\gamma] \subset \mathfrak{g}_e$ and $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$ is a symmetric Lie algebra. Endowed with the inner product $\tilde{B}$, the Lie algebra $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$ is an orthogonal symmetric Lie algebra. The Killing-Cartan form is not degerate on $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$. Then $\mathfrak{g}_e \oplus \mathfrak{g}_\gamma$ is semi-simple. It is a direct sum of orthogonal symmetric Lie algebras of the following two kinds:

$$\begin{array}{ll} i) & \mathfrak{g} = \mathfrak{g}' + \mathfrak{g}' \text{with } \mathfrak{g}' \text{ simple} \\ ii) & \mathfrak{g} \text{ is simple.} \end{array}$$

The first case has been study above and the representation is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-irreducible. In the second case $ad\,[\mathfrak{g}_\gamma, \mathfrak{g}_\gamma]$ is irreducible in $\mathfrak{g}_\gamma$ and the representation is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-irreducible on $\mathfrak{m}$.

# 4 Flag manifolds

In this section we study riemannian properties of the oriented flag manifold

$$M = SO(2l+1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)$$

associated to its $\Gamma$-symmetric structures.

For $\mathfrak{g}$ classical complex simple Lie algebra of type $B_l$, it is always possible to endow $\mathfrak{g}$ with a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading such that

$$\mathfrak{g}_e = so(r_1) \oplus ... \oplus so(r_4)$$

with $r_1 + r_2 + r_3 + r_4 = 2l + 1$ [1]. The compact homogeneous space

$$M = SO(2l+1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)$$

is a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space. We suppose $r_1 \leq r_2 \leq r_3 \leq r_4$. In case $r_1 r_2 \neq 0$ and $r_3 = r_4 = 0$ then $M$ is a symmetric space. The symmetric structure on the Grasmannian

$$SO(2l+1)/SO(r_1) \times SO(r_2)$$

is well known (see [4]). If $r_1r_2r_3 \neq 0$, then the homogeneous space $M$ can not be symmetric. In what follows we shall explicitly construct on $M$ a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-riemannian structure. Let us consider the decomposition of a matrix of $so(2l+1)$

$$\left(\begin{array}{c|c|c|c} X_1 & A_1 & B_1 & C_1 \\ \hline -\,{}^tA_1 & X_2 & C_2 & B_2 \\ \hline -\,{}^tB_1 & -\,{}^tC_2 & X_3 & A_2 \\ \hline -\,{}^tC_1 & -\,{}^tB_2 & -\,{}^tA_2 & X_4 \end{array}\right)$$

with $A_1 \in \mathcal{M}(r_1, r_2)$, $B_1 \in \mathcal{M}(r_1, r_3)$, $C_1 \in \mathcal{M}(r_1, r_4)$, $C_2 \in \mathcal{M}(r_2, r_3)$, $B_2 \in \mathcal{M}(r_2, r_4)$, $A_2 \in \mathcal{M}(r_3, r_4)$ and $X_i \in so(r_i)$, $i = 1, ..., 4$. Let us consider the subspaces of $\mathfrak{g}$ :

$$\mathfrak{g}_e = \begin{pmatrix} X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ 0 & 0 & X_3 & 0 \\ 0 & 0 & 0 & X_4 \end{pmatrix}, \ \mathfrak{g}_a = \begin{pmatrix} 0 & A_1 & 0 & 0 \\ -\,{}^tA_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_2 \\ 0 & 0 & -\,{}^tA_2 & 0 \end{pmatrix}$$

$$\mathfrak{g}_b = \begin{pmatrix} 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \\ -\,{}^tB_1 & 0 & 0 & 0 \\ 0 & -\,{}^tB_2 & 0 & 0 \end{pmatrix}, \ \mathfrak{g}_c = \begin{pmatrix} 0 & 0 & 0 & C_1 \\ 0 & 0 & C_2 & 0 \\ 0 & -\,{}^tC_2 & 0 & 0 \\ -\,{}^tC_1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading of $so(2l+1)$. This graduation defines the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space

$$(SO(2l+1); SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4), (\mathbb{Z}_2 \times \mathbb{Z}_2)_G).$$

Let $B$ be a $\mathfrak{g}_e$-invariant inner product on $\mathfrak{g}$. By hypothesis $B(\mathfrak{g}_\alpha, g_\beta) = 0$ as soon as $\alpha \neq \beta$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$. This shows that $B$ is written $B = B_{\mathfrak{g}_e} + B_{\mathfrak{g}_a} + B_{\mathfrak{g}_b} + B_{\mathfrak{g}_c}$ where $B_{\mathfrak{g}_\alpha}$ is an inner product on $\mathfrak{g}_\alpha$. The restriction $B_{\mathfrak{g}_e}$ to $\mathfrak{g}_e$ is a biinvariant inner product. If $r_4 > 2$, all the components $so(r_i)$ are simple Lie algebras and $B_{\mathfrak{g}_e}$ is written

$$B_{\mathfrak{g}_e} = a_1K_1 + a_2K_2 + a_3K_3 + a_4K_4$$

where $K_i$ is the Killing-Cartan form of $so(r_i)$. If some components $so(r_i)$ are abelian from the index $i_0$, that is $r_i \leq 2$ for $i \geq i_0$ then $B_{\mathfrak{g}_e}$ is of the form $\Sigma_{j<i_0} a_jK_j \ + q$ where $q$ is a definite positive form on the abelian Lie algebra $\oplus_{j \geq i} so(r_j)$. Let us compute $B_{\mathfrak{g}_a}$. We denote by $A_1$ the subspace of $\mathfrak{g}_a$ whose vectors are

$$\left(\begin{array}{c|c|c|c} 0 & A_1 & 0 & 0 \\ \hline -\,{}^tA_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

In the same manner we define $A_2$, $B_1$, $B_2$, $C_1$ and $C_2$. For $1 \leq i \leq r_1$ and $r_1 + 1 \leq j \leq r_2$, let $A_{ij}$ be the corresponding elementary matrices of $A_1$ that is the $A_{ij} = (a_{rs})$ with $a_{ij} = -a_{ji} = 1$ other coordinates being equal to 0. Similary

$X_{ij}$ denotes the elementary matrices of the diagonal block corresponding to $so(r_1)$, $Y_{ij}$ to $so(r_2)$, $Z_{ij}$ to $so(r_3)$ and $T_{ij}$ to $so(r_4)$. We have

$$\left\{\begin{array}{ll} [X_{ij}, A_{jl}] = A_{il}, & 1 \leq i < j \leq r_1,\ r_1+1 \leq l \leq r_1+r_2, \\ [X_{ij}, A_{il}] = -A_{jl}, & 1 \leq i < j \leq r_1,\ r_1+1 \leq l \leq r_1+r_2, \end{array}\right.$$

and

$$\left\{\begin{array}{ll} [Y_{ij}, A_{lj}] = A_{li}, & r_1+1 \leq i < j \leq r_1+r_2,\ 1 \leq l \leq r_1, \\ [Y_{ij}, A_{li}] = -A_{lj}, & r_1+1 \leq i < j \leq r_1+r_2,\ 1 \leq l \leq r_1. \end{array}\right.$$

The relation

$$B_{\mathfrak{g}_a}([X_{rs}, A_{ij}], A_{ij}) = 0$$

for all $X_{rs} \in so(r_1) \oplus so(r_2)$ implies

$$\left\{\begin{array}{lll} B_{\mathfrak{g}_a}(A_{ij}, A_{lj}) = 0, & i, l \in 1, ..., r_1, i \neq l, & j = r_1+1, ..., r_1+r_2, \\ B_{\mathfrak{g}_a}(A_{ij}, A_{il}) = 0, & i = 1, ..., r_1, & j, l \in r_1+1, ..., r_1+r_2, j \neq l. \end{array}\right.$$

From the identities

$$\left\{\begin{array}{l} B([X_{il}, A_{lj}], A_{ij}) + B(A_{lj}, [X_{il}, A_{ij}]) = 0, \\ B([Y_{ij}, A_{lj}], A_{li}) + B(A_{lj}, [Y_{ij}, A_{li}]) = 0, \end{array}\right.$$

we obtain

$$\left\{\begin{array}{lll} B_{\mathfrak{g}_a}(A_{ij}, A_{ij}) = B_{\mathfrak{g}_a}(A_{lj}, A_{lj}), & i, l = 1, ..., r_1, & j = r_1+1, ..., r_1+r_2, \\ B_{\mathfrak{g}_a}(A_{li}, A_{li}) = B_{\mathfrak{g}_a}(A_{lj}, A_{lj}), & l = 1, ..., r_1, & j, i = r_1+1, ..., r_1+r_2. \end{array}\right.$$

We deduce that all the basis vectors of $A_1$ have the same norm with respect the inner product $B$. From the identity

$$B([X_{ij}, A_{jl}], A_{js}) + B(A_{jl}, [X_{ij}, A_{js}]) = 0$$

$1 \leq i < j \leq r_1, l, s \in [[r_1+1, ..., r_1+r_2]]$, we obtain

$$B(A_{il}, A_{js}) + B(A_{is}, A_{jl}) = 0.$$

Suppose that $r_1 \geq 3$. There exists $r$, $1 \leq r \leq r_1$ which is not equal to $i$ or $j$. In this case we have

$$[X_{ij}, A_{rs}] = 0$$

and

$$B([X_{ij}, A_{jl}], A_{rs}) + B(A_{jl}, [X_{ij}, A_{rs}]) = 0$$

gives

$$B(A_{il}, A_{rs}) = 0$$

for $r \neq i$. This implies that the vectors $A_{ij}$ are pairwise orthogonal as soon as $r_1 > 2$. It remains now to compute $B(A_1, A_2)$. The action of $so(r_1)$ is faithful on $A_1$ and trivial on $A_2$ . Thus the $(ad_so(r_1))$-invariance of $B_{\mathfrak{g}_a}$ implies that

$$B_{\mathfrak{g}_a}(A_1, A_2) = 0.$$

All the previous identities implies, if $r_4 > 2$, that

$$B_{\mathfrak{g}_a} = t_{A_1}\Sigma(\alpha^1_{ij})^2 + t_{A_2}\Sigma(\alpha^2_{ij})^2,$$

where $\{\alpha^1_{ij}, \alpha^2_{ij}\}$ is the dual basis of the basis of $\mathfrak{g}_a$ given respectively by the elementary matrices of $A_1$ and $A_2$ and $t_{A_1} > 0, t_{A_2} > 0$. All these computations can be extended to the other components $\mathfrak{g}_b$ and $\mathfrak{g}_c$.

**Proposition 9** *If $r_4 > 2$, then all $\mathfrak{g}_e$-invariant inner product on $\mathfrak{m} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ is given by*

$$B = t_{A_1}\Sigma(\alpha^1_{ij})^2 + t_{A_2}\Sigma(\alpha^2_{ij})^2 + t_{B_1}\Sigma(\beta^1_{ij})^2 + t_{B_2}\Sigma(\beta^2_{ij})^2 + t_{C_1}\Sigma(\gamma^1_{ij})^2 + t_{C_2}\Sigma(\gamma^2_{ij})^2$$

*where $\{\alpha^1_{ij}, \alpha^2_{ij}, \beta^1_{ij}, \beta^2_{ij}, \gamma^1_{ij}, \gamma^2_{ij}\}$ is the dual basis of the basis of $A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus C_1 \oplus C_2$ given by the elementary matrices and the parameters $t_{A_1}, t_{A_2}, t_{B_1}, t_{B_2}, t_{C_1}, t_{C_2}$ being nonnegative.*

It remains to examinate the particular cases corresponding to some $r_i$ equal to 2 or 1. This imply that $so(r_i)$ is abelian (and not simple).

1. If $r_1 = 2$ and $r_2 = 1$ then $r_3 = r_4 = 1$ and the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading of $so(5)$ is given by

$$so(5) = (so(2) \oplus so(1) \oplus so(1) \oplus so(1)) \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$$

with $\dim\mathfrak{g}_a = 3$, $\dim\mathfrak{g}_b = 3$, $\dim\mathfrak{g}_c = 3$ and the homogeneous space is isomorphic to

$$SO(5)/SO(2).$$

Every $so(2)$-invariant metric on $\mathfrak{m}$ is of type

$$\begin{aligned} B = \quad & t_{A_1}((\alpha^1_{13})^2 + (\alpha^1_{23})^2) + t_{A_2}(\alpha^2_{45})^2 + t_{B_1}((\beta^1_{14})^2 + (\beta^1_{24})^2) + t_{B_2}(\beta^2_{35})^2 \\ & + t_{C_1}((\gamma^1_{15})^2 + (\gamma^1_{25})^2) + t_{C_2}(\gamma^2_{34})^2. \end{aligned}$$

2. If $r_1 = r_2 = r_3 = 2$ and $r_4 = 1$ then $\mathfrak{g} = so(7)$. The corresponding $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space is isomophic to

$$SO(7)/SO(2) \times SO(2) \times SO(2).$$

In this case the relation $B(A_{il}, A_{rs}) = 0$ is not valid. We deduce that every $(so(2) \oplus so(2) \oplus so(2))$-invariant inner product on $\mathfrak{m}$ is written

$$\begin{aligned} B = \quad & t_{A_1}((\alpha^1_{13})^2 + (\alpha^1_{23})^2 + (\alpha^1_{14})^2 + (\alpha^1_{24})^2)) + u_{A_1}(\alpha^1_{13}\alpha^1_{24} - \alpha^1_{14}\alpha^1_{23}) \\ & + t_{A_2}((\alpha^2_{57})^2 + (\alpha^2_{67})^2) + t_{B_1}((\beta^1_{15})^2 + (\beta^1_{25})^2 + (\beta^1_{16})^2 + (\beta^1_{26})^2) \\ & + u_{B_1}(\beta^1_{15}\beta^1_{26} - \beta^1_{25}\beta^1_{26}) + t_{B_2}((\beta^2_{37})^2 + (\beta^2_{47})^2) \\ & + t_{C_1}((\gamma^1_{17})^2 + (\gamma^1_{27})^2) + t_{C_2}(\gamma^2_{36})^2. \end{aligned}$$

The remaining cases correspond to $r_1 = 2, r_2 = r_3 = r_4 = 1$ which is treated in the example, to $r_1 = 2, r_2 = 1, r_3 = r_4 = 0$ and the homogeneous space is $SO(3)/SO(2)$ and it is a symmetric space and to $r_1 = r_2 = r_3 = 1, r_4 = 0$ and

$\mathfrak{g}_e = \{0\}$. So Proposition 7 and the previous results give all the metric on flag manifolds $M$ which provide $M$ with a riemannian $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure. In general, for these metrics the Levi-Civita connection is not adapted to symmetries. This connection corresponds to the canonical torsionfree connection $\overline{\nabla}$ of the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric homogeneous space if and only if the metric is naturally reductive with respect to the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-graduation. Recall that this means that

$$B([X,Y]_{\mathfrak{m}}, Z) + B([X,Z]_{\mathfrak{m}}, Y) = 0.$$

for all $X, Y, Z \in \mathfrak{m}$. Applying this identity to a triple of vectors in $A_1 \times B_1 \times C_2$ more precisely to a triple $(A_{r_1+1,1}, B_{r_2+1,1}, C_{r_1+1,r_2+1})$ we obtain that

$$t_{A_1} = t_{B_1} = t_{C_2}.$$

If we choose good triple in $A_1 \times B_2 \times C_2$ and $A_2 \times B_2 \times C_2$ we find

$$t_{A_1} = t_{B_2} = t_{C_2}$$

and

$$t_{A_2} = t_{B_2} = t_{C_2}.$$

Suppose now that the inner product corresponds to one of the particular cases that is there is $i_0$ such that $r_{i_0} = 2$. Thus in the expression of $B$ some double products appear. For example in the second case, $r_1 = r_2 = r_3 = 2$ and $r_4 = 1$. As we have

$$[B_{2,5}, C_{4,5}] = -A_{2,4}$$

then

$$B(A_{1,3}, [B_{2,5}, C_{4,5}]) + B([A_{1,3}, B_{2,5}], C_{4,5}]) = 0$$

gives

$$B(A_{1,3}, A_{2,4}) = 0$$

that is $u_{A_1} = 0$. In the same way we find that all coefficients $u$ are equal to 0.

**Proposition 10** *Every invariant metric $g$ on $SO(2l+1)/SO(r_1) \times SO(r_2) \times SO(r_3) \times SO(r_4)$ which is adapted to the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure is given by an inner product $B$ on $\mathfrak{m}$ of type*

$$B = t(\Sigma(\alpha^1_{ij})^2 + \Sigma(\alpha^2_{ij})^2 + \Sigma(\beta^1_{ij})^2 + \Sigma(\beta^2_{ij})^2 + \Sigma(\gamma^1_{ij})^2 + \Sigma(\gamma^2_{ij})^2)$$

*with $t > 0$.*

**Example : The homogeneous manifold** $SO(5)/SO(2) \times SO(2) \times SO(1)$

In the previous section we have described the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-graduation of the Lie algebra $so(5)$ and we have computed the $G$-invariant metrics which are adpated to this graduation. Such a metric is given by an inner product $B$ on $so(5)$ which is written

$$B = t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1\alpha_4 - \alpha_2\alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2).$$

**Proposition 11** *Every inner product on $so(5)$ for which the homogeneous components are pairwise orthogonal and which is $ad\mathfrak{g}_e$-invariant is written:*

$$B = q_1 + t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) + u(\alpha_1\alpha_4 - \alpha_2\alpha_3) + v(\beta_1^2 + \beta_2^2) + w(\gamma_1^2 + \gamma_2^2)$$

*where $q_1$ is any inner product on $\mathfrak{g}_e$ and $4t^2 - u^2 > 0,\ t, v, w > 0$. This inner product gives an adapted riemannian metric on $SO(5)/SO(2) \times SO(2)$ if it is equal to*

$$B = q_1 + t(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \beta_1^2 + \beta_2^2 + \gamma_1^2 + \gamma_2^2).$$

**Remarks.** 1) If $q_1 = \omega_1^2 + \omega_2^2 + \omega_3^2$ and $t = 1$, then $-B$ coincides with the Killing-Cartan form of $so(5)$. Its covariant operator $\nabla_1$ satisfies

$$2(\nabla_1)_X Y = -[X, Y].$$

2) Suppose that $g$ is the metric $B$ on $so(5)$ corresponding to the inner product

$$B = \sum_{i=1}^{3} \omega_i^2 + \sum_{i=1}^{4} \alpha_i^2 + \sum_{i=1}^{2} \beta_i^2 + \sum_{i=1}^{2} \gamma_i^2.$$

To simplify the notations, we shall put $E_i = A_i,\ i = 1, 2, 3, 4,\ E_5 = B_1, E_6 = B_2, E_7 = C_1, E_8 = C_2$. Then the sectionnal curvatures at the identity of $\mathfrak{m}$ are given by

$$g(R(X,Y)Y,X)_0 = \frac{1}{4} B([X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}) + B([X,Y]_{\mathfrak{g}_e}, [X,Y]_{\mathfrak{g}_e})$$

and with respect to the orthonormal basis $\{E_i\}_{i=1,\dots,8}$ we obtain

$$\begin{array}{l}
R_{1221} = R_{1331} = 1, R_{1551} = R_{1771} = 1/4 \\
R_{1441} = R_{1661} = R_{1881} = 0 \\
R_{2442} = 1, R_{2552} = R_{2882} = 1/4 \\
R_{2332} = R_{2662} = R_{2772} = 0 \\
R_{3443} = R_{3553} = R_{3883} = 0 \\
R_{3663} = R_{3773} = 1/4 \\
R_{4554} = R_{4774} = 0 \\
R_{4664} = R_{4884} = 1/4 \\
R_{5665} = 1, R_{5775} = R_{5885} = 1/4 \\
R_{6776} = R_{6886} = 1/4 \\
R_{7887} = 1.
\end{array}$$

So the sectional curvature is positive.

3) *On the Ambrose-Singer tensor.*
In [8] the authors classify the homogeneous riemannian spaces using the Ambrose-Singer tensor $T$. The symmetric case corresponds to $T = 0$. The general riemannian homogeneous spaces are classified in 8 categories distinguished by algebraic properties of $T$. For the riemannian nonsymmetric space $M = SO(5)/SO(2) \times SO(2)$, this tensor corresponds to

$$T = \nabla - \overline{\nabla}.$$

If $\{E_i\}_{i=1,..,8}$ is the orthonormal basis defined above, we consider the linear map on $M$ given by

$$c_{12}(T)(X) = \sum_{i=1}^{8} B_{\mathfrak{m}}(T(E_i, E_j), X).$$

As $T(E_i, E_j) = -T(E_j, E_i)$, we have $c_{12}(T)(X) = 0$ and $B_{\mathfrak{m}}(T(X,Y), Z) = -B_{\mathfrak{m}}(T(Y,X), Z)$ and the tensor $T$ is of type $\mathcal{T}_3$ in the terminology of [8].

4) *On the geodesics.*
Following [4], if we set for each $X \in \mathfrak{m} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$, $f_t = exp(tX) \in SO(5)$ and $x_t = f_t(0) \in M = SO(5)/SO(2) \times SO(2) \times SO(1)$ where 0 is the coset $SO(2) \times SO(2) \times SO(1)$ in $M$, then the curve $x_t$ is a geodesic in $M$. Conversely each geodesic starting from 0 is of the form $exp(tX)(0)$ for some $X \in \mathfrak{m}$. It is not hard to see that for $E \in \mathfrak{m}$, where $E$ stands for one of the $A_1, A_2, A_3, A_4, B_1, B_2, C_1$ or $C_2$, then $exp(tE) = (I_8 + E^2 + sintE - costE^2$ where $I_8$ is the identity of rank 8.

Two points $exp(t_1)E$ and $exp(t_2 E)$ of this $2\pi$-periodic curve falls in the same coset of $M$ if and only if $t_2 - t_1 = 2k\pi$ for some $k \in \mathbb{Z}$. This shows that $f_t$ projects in a one-to-one manner in $M$ and its image $x_t$ is a closed geodesic (of lenght $2\pi$).

As an example one has

$$exp(tA_1) = \begin{pmatrix} cost & 0 & sint & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ sint & 0 & cost & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

# 5 On lorentzian $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure

It is easy to generalize the notion of riemannian $\Gamma$-symmetric homogeneous space to the notion of semi-riemannian $\Gamma$-symmetric homogeneous space, in particular to a lorentzian metric. A lorentzian symmetric space $M = G/H$ is determinated by a nondegenerate $ad\mathfrak{h}$-invariant bilinear form on $\mathfrak{m}$ of signature $(1, n-1)$. In this case $M$ the Riemann curvature tensor of the Levi-Civita connection is covariant constant.

**Definition 12** *Let $(G, H, \Gamma_G)$ a $\Gamma$-symmetric space, $g$ a semi-riemannian metric of signature $(1, n-1)$ where $n = dim\, M$ and $B$ the corresponding $ad\mathfrak{g}_e$-invariant symmetric bilinear form on $\mathfrak{m}$ . Then $M = G/H$ is called a $\Gamma$-symmetric lorentzian space if the homogeneous componants of $\mathfrak{m}$ are pairwise orthogonal with respect to $B$.*

Since in the riemannian case, this doesnot imply that the riemannian connection $\nabla_g$ of $g$ coincides with $\overline{\nabla}$. If $g$ satisfies this property, we will say that the connection $\nabla_g$ is adapted to the $\Gamma$-symmetric structure.

From the classification of $ad\mathfrak{g}_e$-invariant form on $so(2l+1)$ given in Proposition 7, the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space $SO(2l+1)/SO(r_1) \times ... \times SO(r_4)$ is lorentzian if and only if there exists one homogeneous component of $\mathfrak{m}$ of one dimensional. For example if we consider the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric space $SO(5)/SO(2) \times SO(2) \times SO(1)$ the homogeneous components are of dimension 2 and every semi-riemannian metric is of signature $(2p, 8-2p)$ and cannot be a lorentzian metric. So $SO(5)/SO(2) \times SO(2) \times SO(1)$ can not be lorentzian. Nevertheless one can consider the grading of $so(5)$ given by

$$\begin{pmatrix} 0 & a_1 & b_1 & b_2 & b_3 \\ -a_1 & 0 & c_1 & c_2 & c_3 \\ -b_1 & -c_1 & 0 & x_1 & x_2 \\ -b_2 & -c_2 & -x_1 & 0 & x_3 \\ -b_3 & -c_3 & -x_2 & -x_3 & 0 \end{pmatrix}$$

where $\mathfrak{g}_e$ is parametrized by $x_1, x_2, x_3$, $\mathfrak{g}_a$ by $a_1$, $\mathfrak{g}_b$ by $b_1, b_2, b_3$ and $\mathfrak{g}_c$ by $c_1, c_2, c_3$. Let us denote by $\{X_1, X_2, X_3, A_1, B_1, B_2, B_3, C_1, C_2, C_3\}$ the corresponding graded basis. Here $\mathfrak{g}_e$ is isomorphic to $so(3) \oplus so(1) \oplus so(1)$ and we obtain the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric homogeneous space

$$SO(5)/SO(3) \times SO(1) \times SO(1) = SO(5)/SO(3).$$

Every nondegenerated symmetric bilinear form on $so(5)$ invariant by $g_e = so(3)$ is written

$$q = t(\omega_1^2 + \omega_2^2 + \omega_3^2) + u\alpha_1^2 + v(\beta_1^2 + \beta_2^2 + \beta_3^2) + w(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)$$

where$\{\omega_i, \alpha_1, \beta_i, \gamma_i\}$ is the dual basis of the basis $\{X_i, A_1, B_i, C_i\}$. In particular

**Proposition 13** *The lorentzian inner product*

$$q = \omega_1^2 + \omega_2^2 + \omega_3^2 - \alpha_1^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$

*induces a structure of lorentzian $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-symmetric structure on the nonsymmetric homogeneous space*

$$SO(5)/SO(3).$$

# References


[1] Bahturin Y., Goze M., Γ-symmetric homogeneous spaces. Prepint Mulhouse (2006).

[2] Kac Victor G., *Infinite-dimensional Lie algebras.* Second edition. Cambridge University Press, Cambridge, 1985.

[3] Kowalski O., *Generalized symmetric spaces.* Lecture Notes in Mathematics, 805. Springer-Verlag, Berlin-New York, 1980.

[4] Kobayashi S., Nomizu K., *Foundations of differential geometry* . Interscience tracts in pure and applied mathematics, Number 15, Volumes I and II, Interscience publishers, 1969.

[5] Lutz R., *Sur la géométrie des espaces* $\Gamma$*-symétriques.* C. R. Acad. Sci. Paris Sr. I Math. 293 (1981), no. 1, 55–5.

[6] Ledger A. J., Obata M., *Affine and Riemannian s-manifolds.* J. Differential Geometry 2 1968 451–459.

[7] Reyes W., *A metric for a flag space.* Analysis, Geometry and Probability (Valpareiso 81), Lecture Notes in Pure and Applied Math, 96, (1985), 173–180.

[8] Tricerri F., Vanhecke L., *Homogeneous structures on Riemannian manifolds.* London Mathematical Society Lecture Note Series, **83**. Cambridge University Press, Cambridge, 1983.